Desingularization of First Order Linear Difference Systems with Rational Function Coefficients

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ABSTRACT

It is well known that for a first order system of linear difference equations with rational function coefficients, a solution that is holomorphic in some left half plane can be analytically continued to a meromorphic solution in the whole complex plane. The poles stem from the singularities of the rational function coefficients of the system. Just as for differential equations, not all of these singularities necessarily lead to poles in solutions, as they might be what is called removable. In our work, we show how to detect and remove these singularities and further study the connection between poles of solutions and removable singularities. We describe two algorithms to (partially) desingularize a given difference system and present a characterization of removable singularities in terms of shifts of the original system.

KEYWORDS

systems of linear difference equations, apparent singularities, desingularization, removable singularities.

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1 INTRODUCTION

First order linear difference systems are a class of pseudo-linear systems [5, 9, 15] of the form $\phi(Y) = AY$, where ϕ is the forward- or backward shift operator and A an invertible matrix with, in our case, rational function coefficients. To study properties of possible solutions Y, it is not always necessary to explicitly compute the solution space, but one can rather obtain the information from the system itself. Properties that can be derived in this fashion comprise, among others, the asymptotic behavior [6, 7, 13], positive/negative (semi-) definiteness [1, 22], holomorphicity, and closure properties of (a class of) solutions [18].

In the center of attention when analyzing difference and differential systems lie the poles of the rational function coefficients. It is well known that, like in the case of differential equations and systems, not all poles of the coefficients of a difference system lead to singularities for solutions. These apparent singularities can therefore distort the properties of solutions and should be circumvented in the analysis. One technique to do so is *desingularization*—transforming a given system (or operator) in a way that removes as many poles of the system as possible to discard apparent singularities. In this paper we describe the first algorithm to desingularize first order linear difference systems with rational function coefficients. Our main tool in the treatment of these systems are polynomial basis transformations. We show how to achieve desingularization by composing several basic and easy to compute transformations, and our procedure results in the provably "smallest possible" such desingularizing transformation in the sense that any other desingularizing transformation can be obtained as a right multiple.

The main contributions of this paper are:

(1) The first algorithm to desingularize—partially, or, if possible, completely—first order linear difference systems with rational function coefficients.
(2) A non-trivial necessary and sufficient condition for a given system to be desingularizable at a given singularity.
(3) With the help of (2), an analysis of the connection between removable and apparent singularities of difference systems and their meromorphic function solutions.
(4) An algorithm for reducing the rank of the leading matrix at a singularity of a linear difference system.

In the context of single linear difference equations [1, 2], linear differential equations [21] and, more general, Ore operators [10, 11, 17], desingularization and the effects of removable singularities have been extensively studied in recent years. In [22], the author presents an extension of the idea of desingularization that also takes into account the leading number coefficients of Ore operators. For first order differential equations, a first algorithm for desingularization was given in [3].

It is possible to convert any first order linear difference system to a difference operator of higher order and vice versa [4, 6–8]. Desingularization of systems could therefore be done by computing for a given system the corresponding operator, use existing techniques to desingularize the operator and then constructing the

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desingularized system from the new operator. While this is possible, the procedure comes with at least two caveats:

(1) It can be observed that the coefficients grow very large in the conversion process which has severe negative impact on the computation time.
(2) Desingularization on operator level is done by finding a suitable left-multiple of the given operator. In general, this leads to an increase in order, and thus to an increase in the dimension of the solution space.

Both problems are avoided when dealing directly with systems instead of operators, making the results presented in this paper an essential tool for analyzing difference systems.

The paper is organized as follows. In Section 2 we remind the reader of the formal definition of linear difference systems with rational function coefficients, well known results about meromorphic function solutions and the notion of apparent singularities. In Section 3, we present an algorithm to remove poles of difference systems and give a necessary and sufficient condition for a singularity to be removable. Lastly, the connection between removable poles and apparent singularities is established in Section 4 before concluding the paper in Section 5.

2 DIFFERENCE SYSTEMS AND REMOVABLE SINGULARITIES

Let C be a subfield of the field $\mathbb{C}$ of complex numbers, $C(z)$ the field of rational functions over C and ϕ the C-automorphism of $C(z)$ defined by $\phi(z) = z+1$. A homogeneous system of first-order linear difference equations with rational function coefficients is a system of the form

$$\phi(Y) = AY, \tag{1}$$

where Y is an unknown d-dimensional column vector, $\phi(Y)$ is defined component-wise, and A is an element of $\mathrm{GL}_d(C(z))$, the group of invertible matrices of size $d \times d$ with entries in $C(z)$. We denote the set of matrices of size $d\times d$ with entries in $C[z]$ as $\mathrm{Mat}_d(C[z])$. A (block) diagonal matrix with entries (respectively blocks) $a_1, \dots, a_d$ is denoted by $\mathrm{diag}(a_1, \dots, a_d)$. We will refer to system (1) as $[A]_\phi$.

Given a matrix $T \in \mathrm{GL}_d(C(z))$, we can apply a basis transformation

$$Y = TX,$$

and substitute TX into system (1) to arrive at an equivalent system

$$\phi(X) = T[A]_\phi X,$$

where $T[A]_\phi$ is defined as

$$T[A]_\phi := \phi(T^{-1})AT.$$

A difference system $[A]_\phi$ can be rewritten as

$$\phi^{-1}(Y) = A^*Y, \tag{2}$$

where $A^* := \phi^{-1}(A^{-1})$. We will refer to system (2) as $[A^*]_{\phi^{-1}}$. A transformation $Y = TX$ yields the equivalent system

$$\phi^{-1}(X) = T[A^*]_{\phi^{-1}}X,$$

with

$$T[A^*]_{\phi^{-1}} := \phi^{-1}(T^{-1})A^*T.$$

The set of meromorphic solutions of $[A]_\phi$ form a vector space of dimension d over the field of 1-periodic meromorphic functions. It is well known [19] that any difference system $[A]_\phi$ possesses a fundamental matrix of meromorphic solutions. If F is a holomorphic solution of (1) in some left half plane ($\mathrm{Re}\, z < \lambda$ for some $\lambda \in \mathbb{R}$), then it can be analytically continued to a meromorphic solution in the whole complex plane $\mathbb{C}$ using the relations:

$$\begin{aligned} F(z) &= \phi^{-1}(A)\phi^{-2}(A)\cdots\phi^{-n}(A)\phi^{-n}(F)(z) \\ &= A(z-1)A(z-2)\cdots A(z-n)F(z-n), \end{aligned}$$

which are valid everywhere except at the points of the form $\zeta + n$ where ζ is a pole of A and n is a positive integer ($n = 1, 2, \dots$). If F is a holomorphic solution of (1) in some right half plane ($\mathrm{Re}\, z > \lambda$), then it can be analytically continued to a meromorphic solution in the whole complex plane $\mathbb{C}$ using the relations:

$$\begin{aligned} F(z) &= \phi(A^*)\phi^2(A^*)\cdots\phi^n(A^*)\phi^n F(z) \\ &= A^*(z+1)A^*(z+2)\cdots A^*(z+n)F(z+n), \end{aligned}$$

which are valid everywhere except at the points of the form $\zeta - n$ where ζ is a pole of A^* and n is a positive integer ($n = 1, 2, \dots$).

We will denote by $\mathcal{P}_r(A)$ (respectively $\mathcal{P}_l(A)$) the set of poles of A (respectively A^*). The elements of $\mathcal{P}_r(A)$ (respectively $\mathcal{P}_l(A)$) will be called the r- (respectively l-) singularities of the system (1). A point $\zeta \in \mathbb{C}$ is said to be *congruent* to a given r- (respectively l-) singularity ζ_0 of $[A]_\phi$ if $\zeta = \zeta_0 + k$ (respectively $\zeta = \zeta_0 - k$) for some positive integer k.

The finite singularities of the solutions of $[A]_\phi$ are among the points that are congruent to the singularities of the system.

Definition 2.1. Let ζ be a pole of A (respectively pole of A^*). It is called

(1) a removable r- (respectively l-) singularity if any solution of $[A]_\phi$ which is holomorphic in some left (respectively right) half-plane can be analytically continued to a meromorphic solution which is holomorphic at $\zeta + 1$ (respectively $\zeta - 1$).
(2) an apparent r- (respectively l-) singularity if any solution of $[A]_\phi$ which is holomorphic in some left (respectively right) half-plane can be analytically continued to a meromorphic solution which is holomorphic at each point of $\zeta + \mathbb{N}^*$ (respectively $\zeta - \mathbb{N}^*$).

Example 2.2. A 2×2 system of linear difference equations is given by

$$Y(z+1) = AY = \begin{pmatrix} 0 & 1 \\ \frac{-2(z+1)}{z-2} & \frac{3(z-1)}{z-2} \end{pmatrix} Y(z), \quad A^* = \begin{pmatrix} \frac{3(z-2)}{2z} & \frac{3-z}{2z} \\ 1 & 0 \end{pmatrix}.$$

Here $\mathcal{P}_r(A) = \{2\}$ and the points that are congruent to $\zeta = 2$ are $3, 4, 5, \dots$. We have $\mathcal{P}_l(A) = \{0\}$ and the corresponding congruent points are $-1, -2, -3, \dots$. It can be easily verified that a fundamental matrix of solutions of this system is given by

$$F(z) = \begin{pmatrix} 2^z & z^3 + 5z + 6 \\ 2^{z+1} & z^3 + 3z^2 + 8z + 12 \end{pmatrix}.$$

We focus on studying r-singularities. L-singularities can be removed in the same way by considering A^* and ϕ^{-1} instead of A and ϕ.

We give an algebraic characterization of removable singularities. Let $q \in C[z]$ be an irreducible polynomial. For $f \in C(z) \setminus \{0\}$, we define $\mathrm{ord}_q(f)$ to be the integer n such that $f = q^n \frac{a}{b}$, with $a, b \in C[z] \setminus \{0\}$, $q \nmid a$ and $q \nmid b$. We put $\mathrm{ord}_q(0) = +\infty$. Let $O_q = \{f \in$

$C(z) : \mathrm{ord}_q(f) \geq 0\}$ be the *local ring* at q and O_q/qO_q the residue field of $C(z)$ at q. Let π_q denote the canonical homomorphism from $C[z]$ onto $C[z]/\langle q\rangle$. It can be extended to a ring-homomorphism from O_q onto $C[z]/\langle q\rangle$ as follows: let $f \in O_q$; by definition of O_q, f can be written $f = a/b$ where $a, b \in C[z]$ and $q \nmid b$. We can find $u, v \in C[z]$ such that $ub + vq = 1$, *the value* of f at q, denoted by $\pi_q(f)$, is then defined as $\pi_q(ua)$. Sometimes we write $f \, \mathbf{mod} \, q$ for $\pi_q(f)$. It is clear that π_q is well-defined on O_q and is a surjective ring-homomorphism. The kernel of π_q is qO_q, so O_q/qO_q and $C[z]/\langle q\rangle$ are isomorphic.

If $A = (a_{i,j})$ is a finite-dimensional matrix with entries in $C(z)$, we define the *order* at q of A by $\mathrm{ord}_q(A) := \min_{i,j}(\mathrm{ord}_q(a_{i,j}))$. We say that A has a pole at q if $\mathrm{ord}_q(A) < 0$. We define the *leading matrix* of A at q (notation $\mathrm{lc}_q(A)$) as the leading coefficient $A_{0,q}$ in the q-adic expansion of A:

$$A = q^{\mathrm{ord}_q(A)}(A_{0,q} + qA_{1,q} + q^2 A_{2,q} + \dots).$$

Here the coefficients $A_{i,q}$ are matrices with entries in the field $C[z]/\langle q\rangle$. Note that the matrix $A_{0,q}$ is the value of the matrix $q^{-\mathrm{ord}_q(A)}A$ at q.

For a rational function $r = p/q$ with p monic and $\gcd(p, q) = 1$, we write $\mathrm{num}(r) := p$ and $\mathrm{den}(r) := q$. Similarly, for a matrix $M \in \mathrm{Mat}_d(C(z))$ we denote by $\mathrm{den}(M)$ the common denominator of all the entries of M and denote by $\mathrm{num}(M)$ the polynomial matrix $\mathrm{num}(M) := \mathrm{den}(M)M$.

Definition 2.3. Let $A \in \mathrm{GL}_d(C(z))$ and $q \in C[z]$ be an irreducible polynomial. We say that q is a ϕ-minimal pole of A if $q \mid \mathrm{den}(A)$ and for all $j \in \mathbb{N}^*$, $\phi^j(q) \nmid \mathrm{den}(A)$.

We can now give an algebraic definition of desingularizability of difference systems in an inductive fashion.

Definition 2.4. Let $A \in \mathrm{GL}_d(C(z))$ and let $q \in C[z]$ be an irreducible pole of A.

(1) If q is ϕ-minimal, we say that the system $[A]_\phi$ is partially desingularizable at q if there exists a polynomial transformation $T \in \mathrm{GL}_d(C(z)) \cap \mathrm{Mat}_d(C[z])$ such that $\mathrm{ord}_q(T[A]_\phi) > \mathrm{ord}_q(A)$ and $\mathrm{ord}_p(T[A]_\phi) \geq \mathrm{ord}_p(A)$ for any other irreducible polynomial $p \in C[z]$. If moreover, $\mathrm{ord}_q(T[A]_\phi) \geq 0$ then we say that $[A]_\phi$ is desingularizable at q and we call T a desingularizing transformation for $[A]_\phi$ at q.

(2) If q is not ϕ-minimal, then we call $[A]_\phi$ (partially) desingularizable at q if there exists a desingularizing transformation T for all poles of A of the form $\phi^k(q)$, $k \geq 1$, and $T[A]_\phi$ is either (partially) desingularized at q or (partially) desingularizable at q.

While it is immediate that, for a ϕ-minimal pole q, the algebraic notion of desingularization implies that the roots of q are removable in the sense of Definition 2.1, the converse is not obvious and is proven later in Section 4. Consequently, the roots of q are apparent singularities if (and only if) the system A is desingularizable at all poles of A of the form $\phi^{-k}(q)$, $k \geq 0$. In practice, in order to desingularize a system at a non-ϕ-minimal pole q, one first removes the ϕ-minimal pole congruent to q. The resulting system then has a new ϕ-minimal pole 'closer' to q. One can repeat this process until q itself is ϕ-minimal and eventually removed. A desingularizing transformation for q is then given by the product of all the transformations obtained during this process.

Let us illustrate in the next example why we require removing all singularities left of a given pole, thus making it ϕ-minimal, before considering it eligible for desingularization.

Example 2.5. The system $[A]_\phi$ given by

$$A = \mathrm{diag}\left(\frac{(z+1)^2}{z}, \frac{1}{z+1}\right),$$

can be transformed via $T = \mathrm{diag}(z, 1)$ to $T[A]_\phi = \mathrm{diag}(z+1, \frac{1}{z+1})$. The transformed systems still does not enable analytic continuation at 0 of solutions that are holomorphic in the left half-plane with $\mathrm{Re}(z) < 0$.

3 REMOVING R-SINGULARITIES

3.1 ϕ-Minimal Desingularization

We begin our discussion of removing r-singularities by deriving a method for shifting a factor in the denominator of a given system in a way that allows, if possible, cancellation with zeroes of the system. For this we bring the leading matrix of A into a specific form.

LEMMA 3.1. *Let A be a $d \times d$ matrix with entries in $C(z)$ and let $q \in C[z]$ be an irreducible pole of A. Set $n := -\mathrm{ord}_q(A)$ and $r := \mathrm{rank}(\mathrm{lc}_q(A))$, the rank of the leading matrix of A at q. There exists a unimodular polynomial transformation S such that $S[A]_\phi$ is of the form*

$$\begin{pmatrix} \frac{1}{q^n}A_1 & \frac{1}{q^{n-1}}A_2, \end{pmatrix}, \tag{3}$$

where A_1, A_2 are matrices with entries in O_q of size $d \times r$ and $d \times d-r$ respectively with $\mathrm{rank}(A_1) = r$.

PROOF. The leading matrix $\mathrm{lc}_q(A)$ of A at q is a matrix with entries in the residue field $C[z]/\langle q\rangle$. There exists a non-singular matrix Q with entries in $C[z]/\langle q\rangle$ such that $\mathrm{lc}_q(A) \cdot Q$ is in a column-reduced form, i.e. the last $d - r$ columns of $\mathrm{lc}_q(A) \cdot Q$ are zero, and Q is of the form $Q = C \cdot (\mathrm{I}_d + U)$, where C is a non-singular constant matrix, I_d the identity matrix of dimension $d \times d$, and U is a strictly upper triangular matrix. Taking $S = Q$ as a matrix in $\mathrm{Mat}_d(C[z])$ will result in $S[A]_\phi$ as desired. □

Example 3.2 (Example 2.2 continued). If we set $q = z - 2$, then the leading matrix of the system in Example 2.2 at q is

$$\mathrm{lc}_q(A) = (qA) \, \mathbf{mod} \, q = \begin{pmatrix} 0 & 0 \\ -6 & 3 \end{pmatrix}.$$

A suitable transformation to bring this matrix into a column-reduced form is

$$S = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

Applying S to A gives

$$S[A]_\phi = \begin{pmatrix} \frac{z+1}{z-2} & 0 \\ \frac{-2z-2}{z-2} & 2 \end{pmatrix}.$$

Lemma 3.3. *Let $A \in \mathrm{GL}_d(C(z))$ and $q \in C[z]$ be a ϕ-minimal pole of A. Suppose that A is of the form* (3) *and let $r = \mathrm{rank}(\mathrm{lc}_q(A))$. If $[A]_\phi$ is partially desingularizable at q then any desingularizing transformation T for $[A]_\phi$ can be written as*

$$T = D \cdot \tilde{T}, \text{ where}$$

$$D = \mathrm{diag}(\underbrace{q, \ldots, q}_{r \text{ times}}, \underbrace{1, \ldots, 1}_{d-r \text{ times}}) \text{ and } \tilde{T} \in \mathrm{GL}_d(C(z)) \cap \mathrm{Mat}_d(C[z]).$$

Proof. Let $n = -\mathrm{ord}_q(A)$. Suppose we are given a desingularizing transformation $T \in \mathrm{GL}_d(C(z))$ and let $B = T[A]_\phi$. Then we have that $\phi(T)B = AT$ and hence

$$\phi(T)(q^n B) = (q^n A)T.$$

Since $\mathrm{ord}_q(q^n B) > 0$ and the orders at q of the other matrices involved in the equality are non-negative, we get

$$\pi_q(q^n A)\pi_q(T) = 0.$$

By assumption, the matrix $\pi_q(q^n A)$ is of the form

$$\pi_q(q^n A) = \begin{pmatrix} A_1 & 0 \end{pmatrix},$$

where A_1 is a $d \times r$ matrix with linearly independent columns. Thus, the first r rows of $\pi_q(T)$ must be zero, i.e the first r rows of T have to be divisible by q. This yields the claim. □

Remark 1. *Note that the determinant of any desingularizing transformation T of A at q, not necessarily ϕ-minimal, is divisible by q; in fact $q^r \mid \det(T)$. It then follows that $\phi(q)$ divides $\det(\phi(T))$; in fact $\phi(q)^r \mid \det(\phi(T))$.*

Lemma 3.4. *Let $A \in \mathrm{GL}_d(C(z))$ and let $q \in C[z]$ with $q \mid \mathrm{den}(A)$ be an irreducible pole. If $[A]_\phi$ is (partially) desingularizable at q then there exists a maximal positive integer ℓ such that $\phi^\ell(q) \mid \mathrm{num}(\det(A))$.*

Proof. First, suppose q is ϕ-minimal. There are only finitely many factors of $\mathrm{num}(\det(A))$ of positive degree because A is non-singular. Thus it suffices to show that there exists a positive integer ℓ_0 such that $\phi^{\ell_0}(q) \mid \mathrm{num}(\det(A))$. Let T be a desingularizing transformation of A at q. Put $B := T[A]_\phi$ and denote $\det(T)$ by t. Then, due to the desingularization property, we have that

$$\phi(T)^{-1}\mathrm{num}(A)T = \frac{\mathrm{den}(A)}{\mathrm{den}(B)}\mathrm{num}(B) \in \mathrm{Mat}_d(C[z]).$$

Hence

$$\frac{\det(\mathrm{num}(A))t}{\phi(t)} \in C[z].$$

Let ℓ_0 be the largest integer such that $\phi^{\ell_0}(q) \mid \phi(t)$. By Remark 1, ℓ_0 is strictly positive. Since $\phi^{\ell_0}(q) \nmid t$, it follows that $\phi^{\ell_0}(q) \mid \det(\mathrm{num}(A))$. Now from the relation $\det(\mathrm{num}(A)) = \mathrm{den}(A)^d \det(A)$ and since we assumed that $\mathrm{den}(A)$ has no factor of the form $\phi^j(q)$ with $j \in \mathbb{N}^*$ we can conclude that $\phi^{\ell_0}(q) \mid \mathrm{num}(\det(A))$. To see that the theorem holds for non-ϕ-minimal poles, let $\tilde{q}$ be a non-ϕ-minimal pole congruent to q, i.e. there exists a positive integer k such that $\phi^k(\tilde{q}) = q$. Then $\phi^{k+\ell_0}(\tilde{q}) = \phi^{\ell_0}(q) \mid \mathrm{num}(\det(A))$. □

Definition 3.5. Let $A \in \mathrm{GL}_d(C(z))$ and $q \in C[z]$ be an irreducible pole of A. We define the ϕ-dispersion of A at q as :

$$\phi\text{-}dispersion(A, q) = \max\{\ell \in \mathbb{N}^* \text{ s.t. } \phi^\ell(q) \mid \mathrm{num}(\det(A))\}.$$

When the latter set is empty we put ϕ-dispersion$(A, q) = 0$.

Note that a necessary condition that $[A]_\phi$ can be (partially) desingularized at q is that ϕ-dispersion$(A, q) > 0$.

Example 3.6 (Example 3.2 continued). The determinant of $S[A]_\phi$ in Example 3.2 is $\frac{2(z+1)}{z-2}$. Therefore the ϕ-dispersion of $S[A]_\phi$ at $q = z - 2$ is equal to 3.

We will now describe an algorithm for desingularizing a given system $[A]_\phi$ at a ϕ-minimal pole q. By repeatedly applying the algorithm to $[A]_\phi$, it is then possible to desingularize the system at all removable singularities. It is sufficient to treat the case where q is a single and simple pole of A (i.e. qA has polynomial entries). This is stated in the following lemma.

Lemma 3.7. *Let $A \in \mathrm{GL}_d(C(z))$ and let $q \in C[z]$ be a ϕ-minimal pole of A. Set $h = \frac{\mathrm{den}(A)}{q}$ so that the matrix $hA = q^{-1}\mathrm{num}(A)$ has a single and simple pole at q. Then the system $[A]_\phi$ is (partially) desingularizable at q if and only if the system $[hA]_\phi$ is desingularizable at q. More precisely, a polynomial matrix $T \in \mathrm{GL}_d(C(z))$ is a desingularizing transformation for $[hA]_\phi$ at q if and only if T (partially) desingularizes $[A]_\phi$ at q.*

Proof. It is a direct consequence of the following (trivial but interesting) property: for all $T \in \mathrm{GL}_d(C(z))$ and $h \in C[z] \setminus \{0\}$, one has $T[(hA)]_\phi = h \cdot (T[A]_\phi)$. □

Remark 2. *With the notation of the above lemma, the ϕ-dispersion of $[hA]_\phi$ at q is greater than or equal to the ϕ-dispersion of $[A]_\phi$ at q, and equality holds if q is ϕ-minimal. It follows from the fact that $\det(hA) = h^d \cdot \det(A)$.*

Lemma 3.8. *Let $A \in \mathrm{GL}_d(C(z))$. Suppose that A has a single, simple, irreducible pole at q. If $[A]_\phi$ is desingularizable at q with ϕ-dispersion ℓ, then there exist a unimodular polynomial matrix S and a diagonal polynomial matrix D such that $(S \cdot D)[A]_\phi$ is either desingularized (with respect to q) or desingularizable at $\phi(q)$ with ϕ-dispersion $\ell - 1$.*

Proof. We first take S as in Lemma 3.1 so that $S[A]_\phi$ has the form

$$S[A]_\phi = \begin{pmatrix} \frac{\tilde{A}_{1,1}}{q} & \tilde{A}_{1,2} \\ \frac{\tilde{A}_{2,1}}{q} & \tilde{A}_{2,2} \end{pmatrix},$$

where the $\tilde{A}_{i,j}$ are blocks with polynomial entries, the diagonal blocks are of size $r = \mathrm{rank}(\mathrm{lc}_q(A))$ and $d - r$ respectively. Take $D = \mathrm{diag}(q\mathrm{I}_r, \mathrm{I}_{d-r})$ as in Lemma 3.3. Then the matrix $B := (S \cdot D)[A]_\phi$ has the form

$$B = \begin{pmatrix} \frac{\tilde{A}_{1,1}}{\phi(q)} & \frac{\tilde{A}_{1,2}}{\phi(q)} \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} \end{pmatrix}.$$

The resulting system $[B]_\phi$ has at worst a simple and single pole at $\phi(q)$ with ϕ-dispersion $\ell - 1$. □

Example 3.9 (Example 3.6 continued). The rank of the leading matrix in Example 3.2 is 1. We apply the transformation

$$D_1 = \begin{pmatrix} z-2 & 0 \\ 0 & 1 \end{pmatrix},$$

to $S[A]_\phi$ of Example 3.6 and arrive at the system

$$(S \cdot D_1)[A]_\phi = \begin{pmatrix} \frac{z+1}{z-1} & 0 \\ -2z-2 & 2 \end{pmatrix}.$$

The determinant of $(S \cdot D_1)[A]_\phi$ is $\frac{2(z+1)}{z-1}$. The new ϕ-dispersion is 2.

Theorem 3.10. *Let A be desingularizable at a single, simple, irreducible pole q. Then there exists an integer n, unimodular polynomial matrices $S_1, \ldots, S_n$ and diagonal polynomial matrices $D_1, \ldots, D_n$ such that*

$$T = S_1 \cdot D_1 \cdots S_n \cdot D_n,$$

is a desingularizing transformation for A at q. Furthermore, any other desingularizing transformation T' for A at q can be written as

$$T' = T \cdot \tilde{T} \text{ with } \tilde{T} \in \mathrm{GL}_d(C(z)) \cap \mathrm{Mat}_d(C[z]). \tag{4}$$

Proof. By Lemma 3.4, a desingularizable system $[A]_\phi$ has strictly positive ϕ-dispersion ℓ. Applying the transformation $S \cdot D$ as in Lemma 3.8 gives a system equivalent to $[A]_\phi$ having at worst a pole at $\phi(q)$ (instead of q) but with reduced ϕ-dispersion. After at most ℓ such transformations, the resulting matrix $T[A]_\phi$ has to be desingularized at q. This shows that T can be chosen as in the statement of the theorem. To see that any other desingularizing transformation T' of $[A]_\phi$ at q can be written as in (4), we first note that since S_1 is unimodular, for any such T' we have

$$T' = S_1 \cdot \underbrace{(S_1^{-1} \cdot T')}_{=:T'' \in \mathrm{GL}_d(C(z)) \cap \mathrm{Mat}_d(C[z])},$$

and therefore we can assume that A is of the form (3). Then, as was shown in Lemma 3.3, we can write

$$T'' = D_1 \cdot \tilde{T},$$

with $\tilde{T} \in \mathrm{GL}_d(C(z)) \cap \mathrm{Mat}_d(C[z])$). Again, we can repeat this reasoning n times until we arrive at the desired form. □

Example 3.11 (Example 3.9 continued). The leading matrix of $(S \cdot D_1)[A]_\phi$ as in Example 3.9 at $\phi(q) = z-1$ is already in column-reduced form and of rank 1. We apply the transformation $D_2 = \mathrm{diag}(z-1, 1)$, and get

$$(S \cdot D_1 \cdot D_2)[A]_\phi = \begin{pmatrix} \frac{z+1}{z} & 0 \\ -2z^2+2 & 2 \end{pmatrix}.$$

Again, the leading matrix of this system at $\phi^2(q) = z$ is column-reduced and of rank 1. Finally, after applying the transformation $D_3 = \mathrm{diag}(z, 1)$, we get the desingularized system

$$(S \cdot D_1 \cdot D_2 \cdot D_3)[A]_\phi = \begin{pmatrix} 1 & 0 \\ -2z^3+2z & 2 \end{pmatrix}.$$

Collecting all the transformations, we see that a desingularizing transformation for A at $q = z-2$ is given by

$$T = S \cdot D_1 \cdot D_2 \cdot D_3 = \begin{pmatrix} z^3 - 3z^2 + 2z & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

As was already shown in Lemma 3.4, a positive ϕ-dispersion is a necessary condition for a removable singularity. For a given system $[A]_\phi$ and an irreducible polynomial q, the ϕ-dispersion can be obtained by computing the largest integer root of the resultant $\mathrm{res}_z(q(z+k), \mathrm{num}(\det(A)))$. This, together with Theorem 3.10 and its proof gives rise to Algorithm 1.

Algorithm 1: desingularize_A(A, q)

Input: A with entries in $C(z)$ and a single, simple, irreducible pole $q \in C[z]$.
Output: $(T, T[A]_\phi)$ s.t. $T[A]_\phi$ is desingularized at q, or (I_d, A) if desingularization is not possible.

```
1     T ← I_d
2     WHILE (ϕ-dispersion(A, q) > 0 AND den(A) = 0 mod q) DO
2.1       A_0 ← lc_q(A)
2.2       S ← as in the proof of Lemma 3.1.
2.3       D ← diag(q, ..., q, 1, ..., 1) with rank(A_0) many
              elements equal to q.
2.4       A ← ϕ(S · D)^{-1} · A · (S · D)
2.5       T ← T · S · D
2.6       q ← ϕ(q)
3     IF (den(A) = 0 mod q) RETURN (I_d, A)
4     ELSE RETURN (T, A)
```

3.2 Characterization of Desingularizable Poles

We can give a necessary and sufficient condition for a pole to be desingularizable. It can be seen as the shift analogue of the nilpotency of the leading matrix at the considered pole of the system, which is a necessary condition for an apparent singularity in the differential setting [3].

Proposition 3.12. *Let $q \in C[z]$ be a ϕ-minimal pole of the system $[A]_\phi$. Let $\tilde{A} = q^n A$, so that $\mathrm{ord}_q(\tilde{A}) = 0$ and $\pi_q(\tilde{A}) = \mathrm{lc}_q(A)$. If A is (partially) desingularizable at q then there exists a positive integer k such that*

$$\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A})) = 0. \tag{5}$$

Proof. Let T be a desingularizing transformation for $[A]_\phi$ at q and $B = T[A]_\phi$. Then for all non-negative integers k one has

$$\phi(T)B\phi^{-1}(B) \ldots \phi^{-k}(B) = A\phi^{-1}(A) \ldots \phi^{-k}(A)\phi^{-k}(T),$$

and hence

$$\phi(T)(q^n B)\phi^{-1}(q^n B) \ldots \phi^{-k}(q^n B) = \tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A})\phi^{-k}(T).$$

As $\mathrm{ord}_q(q^n B) > 0$ and

$$\mathrm{ord}_q(\phi^{-j}(q^n B)) = \mathrm{ord}_{\phi^j(q)}(B) \geq \mathrm{ord}_{\phi^j(q)}(A) \geq 0, \text{ for all } j \in \mathbb{N}^*,$$

we get that

$$\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A})\phi^{-k}(T)) = 0.$$

Now we conclude by remarking that for k large enough $\pi_q(T(z-k))$ is invertible. □

We will now show that the factorial relation (5) is a sufficient condition for a matrix A to be partially desingularizable at q.

Proposition 3.13. *Let $q \in C[z]$ be a ϕ-minimal pole of $[A]_\phi$. Let $\tilde{A} = q^n A$, so that $\mathrm{ord}_q(\tilde{A}) = 0$ and $\pi_q(\tilde{A}) = \mathrm{lc}_q(A)$. If $[A]_\phi$ is such that the factorial relation (5) holds for some integer $k \geq 1$ then $[A]_\phi$ is (partially) desingularizable at q.*

Proof. Let k be minimal so that (5) holds. Put

$$M := \pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \cdots \phi^{-k+1}(\tilde{A})) \quad \text{and} \quad N := \pi_q(\phi^{-k}(\tilde{A})).$$

By definition of k, the matrix M is nonzero (but singular) and we have $M \cdot N = 0$. With $d := \dim(A)$ it follows that

$$0 < \mathrm{rank}(M) \leq s := d - \mathrm{rank}(N) < d.$$

Let $P \in \mathrm{GL}_d(C[z]/\langle q \rangle)$ such that $P \cdot N$ has its last $(d-s)$ rows linearly independent over $C[z]/\langle q \rangle$ while its s first rows are zero. Consider the matrix : $U = \phi^{k-1}(P^{-1})$ as an element of $\mathrm{Mat}_d(C[z])$ then by applying the unimodular transformation $Y = UX$, we can assume that the matrix N has the following form:

$$N = \begin{pmatrix} \mathrm{O}_s & \mathrm{O}_{s,d-s} \\ N_{2,1} & N_{2,2} \end{pmatrix},$$

where $N_{2,1}$ and $N_{2,2}$ are matrices with entries in $C[z]/\langle q \rangle$ of size $(d-s) \times s$ and $(d-s) \times (d-s)$ respectively, so that the last $d-s$ rows of N are linearly independent over $C[z]/\langle q \rangle$. As $M \cdot N = 0$ we have that the $d-s$ last columns of M are zero. Let $\tilde{A} = (\tilde{A}_{i,j})_{1 \leq i,j \leq 2}$ be partitioned in four blocks as N. Then $\pi_q(\phi^{-k}(\tilde{A}_{1,j})) = 0$ for $j = 1, 2$. In other words, the s first rows of $\tilde{A}$ are divisible by $\phi^k(q)$. Using the substitution $Y = DX$ where $D = \mathrm{diag}(\phi^{k-1}(q)\mathrm{I}_s, \mathrm{I}_{d-s})$, we get a new system which still has a pole at q of multiplicity at most n. Indeed, we have

$$\begin{aligned} B := \phi(D)^{-1}AD = q^{-n} \begin{pmatrix} \frac{\phi^{k-1}(q)\tilde{A}_{1,1}}{\phi^k(q)} & \frac{\tilde{A}_{1,2}}{\phi^k(q)} \\ \phi^{k-1}(q)\tilde{A}_{2,1} & \tilde{A}_{2,2} \end{pmatrix} = \\ q^{-n} \begin{pmatrix} \phi^{k-1}(q)\tilde{A}'_{1,1} & \tilde{A}'_{1,2} \\ \phi^{k-1}(q)\tilde{A}_{2,1} & \tilde{A}_{2,2} \end{pmatrix}, \end{aligned} \tag{6}$$

for some matrices $\tilde{A}'_{1,1}$, $\tilde{A}'_{1,2}$ with entries in O_q. It is clear that $\mathrm{den}(B) \mid \mathrm{den}(A)$ and that $\mathrm{ord}_q(B) \geq \mathrm{ord}_q(A)$. Now we will prove that the factorial relation (5) holds for $\tilde{B} := q^n B$ with $k-1$ instead of k. For this we remark first that

$$\phi(D)\tilde{B}\phi^{-1}(\tilde{B}) \cdots \phi^{-k+1}(\tilde{B}) = \tilde{A}\phi^{-1}(\tilde{A}) \cdots \phi^{-k+1}(\tilde{A})\phi^{-k+1}(D).$$

It then follows that

$$\pi_q(\phi(D))\pi_q(\tilde{B}\phi^{-1}(\tilde{B}) \cdots \phi^{-k+1}(\tilde{B})) = M \cdot \pi_q(\phi^{-k+1}(D)).$$

We have that

$$\pi_q(\phi^{-k+1}(D)) = \pi_q(\mathrm{diag}(q\mathrm{I}_s, \mathrm{I}_{d-s})) = \mathrm{diag}(\mathrm{O}_s, \mathrm{I}_{d-s}),$$

hence $M \cdot \pi_q(\phi^{-k+1}(D)) = 0$ (since the $d-s$ last columns of M are zero). Now $\pi_q(\phi(D)) = \pi_q(\mathrm{diag}(\phi^k(q)\mathrm{I}_s, \mathrm{I}_{d-s}))$ is invertible (since q and $\phi^k(q)$ are co-prime), it then follows that

$$\pi_q(\tilde{B}\phi^{-1}(\tilde{B}) \cdots \phi^{-k+1}(\tilde{B})) = 0.$$

If $k-1$ is still positive then we can repeat this process for the matrix B and the polynomial q until we arrive at $k = 1$. When $k = 1$ the above factorial relation reduces to $\pi_q(\tilde{B}) = 0$ which means that $\mathrm{ord}_q(\tilde{B}) > 0$ and therefore $\mathrm{ord}_q(B) \geq -n+1$. □

This proof motivates the following alternative desingularization algorithm. In contrast to Algorithm 1, instead of shifting a singularity towards a zero of the system, it performs the analogous task of moving a zero towards the singularity until they cancel each other

Algorithm 2: desingularize_B(A, q)

Input: A with entries in $C(z)$ and a single, simple, irreducible pole $q \in C[z]$.

Output: $(T, T[A]_\phi)$ s.t. $T[A]_\phi$ is desingularized at q.

$T \leftarrow I_d$

WHILE $(\mathrm{den}(A) = 0 \ \mathbf{mod}\ q)$ **DO**

  $\ell \leftarrow \phi - \mathrm{dispersion}(A, q)$

  IF $(\ell \leq 0)$ **THEN RETURN** (T, A)

  $n \leftarrow \mathrm{ord}_q(A)$; $\tilde{A} \leftarrow q^n A$

  $k \leftarrow 0$; $M \leftarrow I_d$; $N \leftarrow \pi_q(\tilde{A})$

  WHILE $(M \cdot N \neq 0$ **AND** $k \leq \ell)$ **DO**

    $M \leftarrow M \cdot N$; $k \leftarrow k+1$; $N \leftarrow \pi_q(\phi^{-k}(\tilde{A}))$

  $U \leftarrow$ as in the proof of Proposition 3.12.

  $D \leftarrow \mathrm{diag}(\phi^{k-1}(q)\mathrm{I}_s, \mathrm{I}_{d-s})$ with $s = d - \mathrm{rank}(N)$.

  $A \leftarrow \phi(U \cdot D)^{-1} \cdot A \cdot (U \cdot D)$

  $T \leftarrow T \cdot U \cdot D$

RETURN (T, A)

An implementation of Algorithm 1 and Algorithm 2 in the computer algebra system Sage [12] can be obtained from

http://www.mjaroschek.com/systemdesing/

Remark 3. *All systems that are desingularizable via Algorithm 1 are also desingularizable via Algorithm 2 and vice versa.*

Example 3.14. For A as in Example 2.2 and $q = z-2$ we have

$$\tilde{A} = (z-2)A = \begin{pmatrix} 0 & z-2 \\ -2(z+1) & 3(z-1) \end{pmatrix},$$

$$M = \pi_q(\tilde{A}(z)\tilde{A}(z-1)\tilde{A}(z-2)) = \tilde{A}(2)\tilde{A}(1)\tilde{A}(0) = \begin{pmatrix} 0 & 0, \\ -12 & 6 \end{pmatrix} \neq 0$$

$$N = \pi_q(\phi^{-3}(\tilde{A}) = \tilde{A}(-1) = \begin{pmatrix} 0 & -3 \\ 0 & -6 \end{pmatrix},$$

$$\pi_q(\tilde{A}(z)\tilde{A}(z-1)\tilde{A}(z-2)\tilde{A}(z-3)) = \tilde{A}(2)\tilde{A}(1)\tilde{A}(0)\tilde{A}(-1) = 0,$$

so $k = 3$. If we chose

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

then

$$\phi(U)^{-1}AU = U^{-1}AU = \frac{1}{(z-2)} \begin{pmatrix} (z+1) & 0 \\ -(z+1) & 2(z-2) \end{pmatrix}.$$

We have $s = 1$, so with $D = \mathrm{diag}(\phi^2(q), 1) = \mathrm{diag}(z, 1)$ we get

$$B = \phi(D)^{-1}(\phi(U)^{-1}AU)D = \frac{1}{(z-2)} \begin{pmatrix} z & 0 \\ -z(z+1) & 2(z-2) \end{pmatrix}.$$

Note that, as expected, we have that

$$\tilde{B}(2)\tilde{B}(1)\tilde{B}(0) = 0.$$

Here we can repeat the above process on B to desingularize as much as possible the matrix A at $q = z-2$. In this particular example q is removable by the transformation $T = U \cdot \mathrm{diag}(z(z-1)(z-2), 1)$. Indeed, one can see that

$$T[A]_\phi = \phi(T)^{-1}AT = \begin{pmatrix} 1 & 0 \\ -z(z^2-1) & 2 \end{pmatrix},$$

has polynomial entries. The transformation T is the same as in Example 3.11 up to a right factor $\mathrm{diag}(\frac{1}{2}, 1)$.

3.3 Rank Reduction

Consider a system $[A]_\phi$ and let q be a ϕ−minimal factor of $\mathrm{den}(A)$ with multiplicity $n \geq 1$, such that $[A]_\phi$ the is not partially desingularizable at q. This implies that there's no positive integer k such that relation (5) holds. As the quantity n cannot be reduced, it's natural to ask if it is possible to reduce the rank of the leading matrix $\mathrm{lc}_q(A)$ by applying a polynomial transformation T to $[A]_\phi$. We shall give a criterion for the existence of a polynomial transformation T such $ord_q(T[A]_\phi) = ord_q(A)$ and $\mathrm{rank}(\mathrm{lc}_q(T[A]_\phi)) < \mathrm{rank}(\mathrm{lc}_q(A))$

Proposition 3.15. *Let $q \in C[z]$ be a ϕ-minimal pole of $[A]_\phi$. Let $\tilde{A}(z) = q^n A(z)$, so that $\mathrm{ord}_q(\tilde{A}) = 0$ and $\pi_q(\tilde{A}) = \mathrm{lc}_q(A)$. Then a necessary and sufficient condition for the existence of a polynomial transformation T such that $ord_q(T[A]) = ord_q(A)$ and $\mathrm{rank}(\mathrm{lc}_q(T[A])) < \mathrm{rank}(\mathrm{lc}_q(A))$ is that there exists a positive integer k such that*

$$\mathrm{rank}(\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A}))) < \mathrm{rank}(\mathrm{lc}_q(A)). \qquad (7)$$

Proof. *Necessary condition*: Suppose first that there exists a polynomial matrix T with the desired properties and let $B = T[A]_\phi$. Similarly to the proof of Proposition 3.12, one gets for all non-negative integers k:

$$\phi(T)(q^n B)\phi^{-1}(q^n B) \ldots \phi^{-k}(q^n B) = \tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A})\phi^{-k}(T).$$

Since $\mathrm{ord}_q(q^n B) = 0 = \mathrm{ord}_q(\tilde{A})$ and all the other factors in both sides of this equality have non-negative orders at q we get that

$$\pi_q(\phi(T))\pi_q((q^n B))\pi_q(\phi^{-1}(q^n B) \ldots \phi^{-k}(q^n B)) = \pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A}))\pi_q(\phi^{-k}(T)).$$

By using the fact that the rank of a product of matrices is less or equal to the rank of each factor we get that the rank of the product in the right hand side of the previous equality is bounded by $\mathrm{rank}(\pi_q((q^n B))) = \mathrm{rank}(\mathrm{lc}_q(B))$ and hence

$$\mathrm{rank}(\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A}))\pi_q(\phi^{-k}(T))) \leq \mathrm{rank}(\mathrm{lc}_q(B)) < \mathrm{rank}(\mathrm{lc}_q(A)).$$

Now let k be the smallest positive integer such that the matrix $\pi_q(T(z-k))$ is of full rank. Then

$$\mathrm{rank}(\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A}))) = \mathrm{rank}(\pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \ldots \phi^{-k}(\tilde{A}))\pi_q(\phi^{-k}(T))) < \mathrm{rank}(\mathrm{lc}_q(A)).$$

Sufficient condition: Let $r = \mathrm{rank}(\mathrm{lc}(A))$ and let k be minimal so that (7) holds. Put

$$M := \pi_q(\tilde{A}\phi^{-1}(\tilde{A}) \cdots \phi^{-k+1}(\tilde{A})) \quad \text{and} \quad N := \pi_q(\phi^{-k}(\tilde{A})).$$

By definition of k, the matrix M is nonzero, has the same rank r as $\mathrm{lc}_q(A)$ and we have the strict inequality

$$\mathrm{rank}(M \cdot N) < r = \mathrm{rank}(M).$$

This implies in particular that $\mathrm{rank}(N) < d = \dim(A)$. Let $s := d - \mathrm{rank}(N)$. As in the proof of Proposition 3.13, we can assume that N has the following form:

$$N = \begin{pmatrix} \mathrm{O}_s & \mathrm{O}_{s,d-s} \\ N_{2,1} & N_{2,2} \end{pmatrix},$$

where $N_{2,1}$ and $N_{2,2}$ are matrices with entries in $C[z]/\langle q \rangle$ of size $(d-s) \times s$ and $(d-s) \times (d-s)$ respectively, so that the last $d-s$ rows of N are linearly independent over $C[z]/\langle q \rangle$. Let $M = (M_{i,j})_{1 \leq i,j \leq 2}$ be partitioned in four blocks as N. Then we have

$$M \cdot N = \begin{pmatrix} M_{1,2} \\ M_{2,2} \end{pmatrix} \cdot \begin{pmatrix} N_{2,1} & N_{2,2} \end{pmatrix}.$$

As the matrix $(N_{2,1} \ \ N_{2,2})$ is of full rank, we get that

$$\mathrm{rank}\begin{pmatrix} M_{1,2} \\ M_{2,2} \end{pmatrix} = \mathrm{rank}(M \cdot N) < r.$$

Let $\tilde{A} = (\tilde{A}_{i,j})_{1 \leq i,j \leq 2}$ be partitioned in four blocks as N. Then $\pi_q(\phi^{-k}(\tilde{A}_{1,j})) = 0$ for $j = 1, 2$. Using the substitution $Y = DX$ where $D = \mathrm{diag}(\phi^{k-1}(q)\mathrm{I}_s, \mathrm{I}_{d-s})$, we get a system $[B]_\phi$ of the form (6). with $\mathrm{den}(B) \mid \mathrm{den}(A)$ and $\mathrm{ord}_q(B) \geq \mathrm{ord}_q(A)$. Note that

$$\pi_q(q^n B) = \begin{pmatrix} \pi_q(\frac{1}{\phi^k(q)})\mathrm{I}_s & \mathrm{O}_{s,d-s} \\ \mathrm{O}_{d-s,s} & \mathrm{I}_{d-s} \end{pmatrix} \cdot \begin{pmatrix} \pi_q(\tilde{A}_{1,1}) & \pi_q(\tilde{A}_{1,2}) \\ \pi_q(\tilde{A}_{2,1}) & \pi_q(\tilde{A}_{2,2}) \end{pmatrix} \cdot \begin{pmatrix} \pi_q(\phi^{k-1}(q))\mathrm{I}_s & \mathrm{O}_{s,d-s} \\ \mathrm{O}_{d-s,s} & \mathrm{I}_{d-s} \end{pmatrix}.$$

It follows that if $k \geq 2$, then $\mathrm{rank}(\pi_q(q^n B)) = \mathrm{rank}(\mathrm{lc}_q(A))$, but we will prove that the factorial relation (7) holds for $\tilde{B} := q^n B$ with $k-1$ instead of k. As in the proof of Proposition 3.13, we have that

$$\pi_q(\phi^{-k+1}(D)) = \pi_q(\mathrm{diag}(q\mathrm{I}_s, \mathrm{I}_{d-s})) = \mathrm{diag}(\mathrm{O}_s, \mathrm{I}_{d-s}),$$

hence

$$M \cdot \pi_q(\phi^{-k+1}(D)) = \begin{pmatrix} \mathrm{O}_s & M_{1,2} \\ \mathrm{O}_{d-s} & M_{2,2} \end{pmatrix},$$

whose rank is less than r. Now $\pi_q(\phi(D)) = \mathrm{diag}(\pi_q(\phi^k(q))\mathrm{I}_s, \mathrm{I}_{d-s})$ is invertible (since q and $\phi^k(q)$ are co-prime), it then follows that

$$\mathrm{rank}(\pi_q(\tilde{B}\phi^{-1}(\tilde{B}) \cdots \phi^{-k+1}(\tilde{B}))) = \mathrm{rank}(M \cdot \pi_q(\phi^{-k+1}(D)) < r = \mathrm{rank}(\mathrm{lc}_q(B)).$$

If $k-1$ is still positive then we can repeat this process on the matrix B and the polynomial q until we arrive at $k = 1$. Then we have that

$$\pi_q(q^n B) = \begin{pmatrix} \pi_q(\frac{1}{\phi(q)})\mathrm{I}_s & \mathrm{O}_{s,d-s} \\ \mathrm{O}_{d-s,s} & \mathrm{I}_{d-s} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{O}_s & M_{1,2} \\ \mathrm{O}_{d-s} & M_{2,2} \end{pmatrix},$$

whose rank is less than r. □

The proof of Proposition 3.15 suggests that Algorithm 2 can be easily adapted to minimize the rank of the leading matrix of a ϕ-minimal pole. In particular, a T can be computed such that $\mathrm{ord}_p(T[A]_\phi) \geq \mathrm{ord}_p(A)$ for $p \in C[z]$. It is to note that rank reduction for a pole in A via Algorithm 2 comes at the potential cost of an increase in order of a pole of A^*, as the next example shows.

Example 3.16. Consider the system with

$$A = \begin{pmatrix} z(z+1) & 0 & 0 \\ 0 & \frac{z+1}{z} & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}, \qquad A^* = \begin{pmatrix} \frac{1}{z(z-1)} & 0 & 0 \\ 0 & \frac{z-1}{z} & 0 \\ 0 & 0 & z-1 \end{pmatrix}.$$

We have $\operatorname{rank}(\operatorname{lc}_z(A)) = 2$ and $\operatorname{ord}_{z-1}(A^*) = -1$, and computing a rank reducing transformation for $[A]_\phi$ via Algorithm 2 gives $T = \operatorname{diag}(z, z, 1)$, which results in

$$T[A]_\phi = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}, \qquad T[A]_\phi^* = \begin{pmatrix} \frac{1}{(z-1)^2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & z-1 \end{pmatrix},$$

with $\operatorname{rank}(\operatorname{lc}_z(T[A]_\phi)) = 1$ and $\operatorname{ord}_{z-1}(A^*) = -2$. We note that we merely shifted an already present pole in A^* to the right, as opposed to adding a new factor to the system.

4 APPARENT SINGULARITIES

In this section we establish the connection between the analytical notion of apparent and removable singularities of meromorphic solutions and the algebraic concept desingularization of difference systems. The key observation is the fact that the factorial relation (5) provides a sufficient condition for a singularity to be removable..

Proposition 4.1. *Let $\zeta \in \mathcal{P}_r(A)$ be a pole of A of order $\nu \geq 1$ such that $\zeta - j \notin P_r(A)$ for all positive integers j. Let $\tilde{A} = (z-\zeta)^\nu A$, so that $\tilde{A}(\zeta) \neq 0$. If ζ is a removable r-singularity of $[A]_\phi$, then there exists a positive integer k such that*

$$\tilde{A}(\zeta)A(\zeta - 1)\cdots A(\zeta - k) = 0.$$

In particular, the matrix $A(\zeta - j)$ is singular for some non-negative integer j.

Proof. Using a result due to Ramis [4, 14, 20], one can easily prove that for any complex number η with $-\operatorname{Re}\eta$ large enough, there exist a meromorphic fundamental matrix solution $F(z)$ which is holomorphic for $-\operatorname{Re} z$ large enough and satisfies $F(\eta) = \mathrm{I}_d$. Choose a positive integer k such that $-\operatorname{Re}(\zeta - k)$ is large enough and take a fundamental matrix solution $F(z)$ as above with $F(\zeta - k) = \mathrm{I}_d$. Then one can write

$$F(z+1) = A(z)A(z-1)A(z-2)\cdots A(z-k)F(z-k),$$

and hence

$$(z-\zeta)^\nu F(z+1) = (z-\zeta)^\nu A(z)A(z-1)A(z-2)\cdots A(z-k)F(z-k).$$

Taking the limit as z goes to ζ, we get that

$$0 = \tilde{A}(\zeta)A(\zeta-1)A(\zeta-2)\cdots A(\zeta-k). \qquad \square$$

Corollary 4.2. *Let $\zeta \in P_r(A)$ such that there is a ϕ-minimal q with $q(\zeta) = 0$. If ζ is a removable singularity of $[A]_\phi$, then $[A]_\phi$ is desingularizable at q.*

Proof. Let $n := -\operatorname{ord}_q(A)$. We can apply Proposition 3.13 to reduce the multiplicity of q in $\operatorname{den}(A)$ from n to $n-1$. If $n > 1$, q is still ϕ-minimal and ζ still removable, and we can repeat the process until $[A]_\phi$ is desingularized at q. $\square$

5 CONCLUSION AND FUTURE WORK

In this paper we presented two algorithms to desingularize linear first order difference systems and we explored the notions of apparent and removable singularities. These topics have already been studied in the context of difference operators, where usually the solution space of a given operator is increased as a side effect of the desingularization process. An interesting starting point for further research is to investigate the relation of desingularization on a system level and on an operator level in regard to this extension of the solution space.

Concerning pseudo linear systems, we will continue our work in several directions. We aim to establish a clear connection between removable singularities of a system $[A]_\phi$ and the removable singularities of $[A^*]_\phi^{-1}$, as well as the role of removable singularities for extending numerical sequences. Furthermore, studying desingularization at non-ϕ-minimal poles is a promising approach for identifying poles that only appear in some components of fundamental solutions. We are currently also investigating how to characterize poles in solutions that do not propagate to infinitely many congruent points via gauge transformations.

Regarding complexity, it would be desirable to conduct a thorough complexity analysis of desingularization algorithms. Finally, as was shown in [10, 16], removable singularities of operators can negatively impact the running time of some algorithms and it is interesting to investigate whether similar effects occur for linear systems.

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